ON PAIRS OF COMMUTING NILPOTENT MATRICES

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Abstract. Let B be a nilpotent matrix and suppose that its Jordan canonical form is determined by a partition λ. Then it is known that its nilpotent commutator $\mathcal{N}_B$ is an irreducible variety and that there is a unique partition μ such that the intersection of the orbit of nilpotent matrices corresponding to μ with $\mathcal{N}_B$ is dense in $\mathcal{N}_B$. We prove that map $\mathcal{D}$ given by $\mathcal{D}(\lambda) = \mu$ is an idempotent map. This answers a question of Basili and Iarrobino [9] and gives a partial answer to a question of Panyushev [18]. In the proof, we use the fact that for a generic matrix $A \in \mathcal{N}_B$ the algebra generated by A and B is a Gorenstein algebra. Thus, a generic pair of commuting nilpotent matrices generates a Gorenstein algebra. We also describe $\mathcal{D}(\lambda)$ in terms of λ if $\mathcal{D}(\lambda)$ has at most two parts.

1. Introduction

We denote by $M_n(F)$ the algebra of all $n \times n$ matrices over an algebraically closed field F and by $\mathcal{N}$ the variety of all nilpotent matrices in $M_n(F)$. Let $B \in \mathcal{N}$ be a nilpotent matrix and suppose that its Jordan canonical form is determined by a partition λ. We denote by $\mathcal{O}_B = \mathcal{O}_\lambda$ the orbit of B under the $\mathrm{GL}_n(F)$ action on $\mathcal{N}$ and by $\mathcal{N}_B$ the nilpotent commutator of B, i.e. the set of all $A \in \mathcal{N}$ such that $AB = BA$. It is known that $\mathcal{N}_B$ is an irreducible variety (see Basili [2]). So there is a unique partition μ of n such that $\mathcal{O}_\mu \cap \mathcal{N}_B$ is dense in $\mathcal{N}_B$. Following Basili and Iarrobino [9], and Panyushev [18] we define a map on the set of all partitions of n by $\mathcal{D}(\lambda) = \mu$. (We note that in [3, 9] this map is denoted by Q.) As in [18] we say that B or its orbit $\mathcal{O}_\lambda$ is *self-large* if $\mathcal{D}(\lambda) = \lambda$. According to [3, 9], a partition λ is called *stable* if $\mathcal{D}(\lambda) = \lambda$.

The work presented in this paper was initially stimulated by a question if the partition $\mathcal{D}(\lambda)$ is stable, i.e. if $\mathcal{D}$ is an idempotent map, that was posed by Basili and Iarrobino in their conference notes [9]. It was later brought to our attention that Panyushev in [18, Problem 2] stated the same question in a more general setup of simple Lie algebras. The general question in the theory of nilpotent orbits of semisimple Lie algebras is similar to the question described above. Suppose that $\mathfrak{g}$ is a semisimple Lie algebra and G its adjoint group. If $x \in \mathfrak{g}$ is a nilpotent element and $\mathfrak{z}_\mathfrak{g}(x)$ its centralizer in $\mathfrak{g}$, then there is a unique maximal nilpotent G-orbit, say Gy, meeting $\mathfrak{z}_\mathfrak{g}(x)$. It was pointed out to us by a referee that the latter follows from the fact that the set $\mathcal{N}_\mathfrak{h}$ of all nilpotent elements of an algebraic Lie algebra $\mathfrak{h}$ is irreducible, which in turn can be deduced from the existence of the Levi decomposition and the irreducibility of $\mathcal{N}_\mathfrak{h}$ in the reductive case, which was proved already by Kostant [10, §3]. The question is then if the largest nilpotent orbit meeting $\mathfrak{z}_\mathfrak{g}(y)$ is Gy itself [18, Problem 2]. A nilpotent orbit Gy is said to be *self-large*, if

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it is the largest nilpotent orbit meeting $\mathfrak{z}_{\mathfrak{g}}(y)$. Thus, the question is if the largest nilpotent orbit meeting $\mathfrak{z}_{\mathfrak{g}}(x)$ is necessarily self-large.

Our main result is the proof of the fact that $\mathcal{D}$ is an idempotent map on nilpotent orbits of $M_n(F)$. This answers the original question of Basili and Iarrobino and hence also a special $\mathfrak{sl}_n$ case of Panyushev's question. In the proof we use an extension of a lemma of Baranovsky [1]. We prove that a generic pair of commuting nilpotent matrices generates a Gorenstein algebra (in fact, a complete intersection [5, Cor. 21.20]). Moreover, a generic matrix $A \in \mathcal{N}_B$ and B generate a Gorenstein algebra. Then we use Macaulay's Theorem on the Hilbert function of the intersection of two plane curves [12] (see also [8]) together with some results of [3] to prove that $\mathcal{D}$ is an idempotent. It appears that an answer to the general question posed by Panyushev [18, Problem 2] requires methods different from ours. We see no immediate generalization of our proof to the general setup of simple Lie algebras.

It is an interesting question [18, Problem 1] to describe $\mathcal{D}(\lambda)$ in terms of partition λ. We would like to mention that some partial results to this question were obtained by the second author in [14, 15]. The main result of [14] (see also [4]) gives the answer in the case when $\mathcal{D}(\lambda)$ has at most two parts. We discuss this in the last section.

Panyushev's work [18] was motivated by Premet's results on the nilpotent commuting variety [21] of a simple Lie algebra. For other results on special pairs of nilpotent commuting elements in these varieties see [6, 17]. Some of other references for the theory of commuting varieties are [16, 19, 20, 22, 23].

This paper is an extension of our unpublished note [11].

2. Gorenstein pairs are dense

In this section we prove that a generic pair of commuting nilpotent matrices generates a Gorenstein (local artinian) algebra.

Suppose that $B \in \mathcal{N}$ and that $A \in \mathcal{N}_B$. Then we denote by $\mathcal{A} = \mathcal{A}(A, B)$ the unital subalgebra of $M_n(F)$ generated by matrices A and B, and by $\mathcal{A}^T = \mathcal{A}(A^T, B^T)$ the unital subalgebra generated by the transposed matrices A^T and B^T. The algebra $\mathcal{A}$ is a commutative local artinian algebra. Such an algebra is Gorenstein if its socle $\mathrm{Soc}(\mathcal{A})$ is a simple $\mathcal{A}$-module (see [5, Prop. 21.5]); i.e., if $\dim_F(\mathrm{Soc}(\mathcal{A})) = 1$, where $\mathrm{Soc}(\mathcal{A}) = (0 : m)$ is the annihilator of the maximal ideal m of $\mathcal{A}$.

We write $\mathcal{N}_2 \subset M_n(F) \times M_n(F)$ for the variety of all commuting pairs of nilpotent matrices. Note that the subset $U \subset \mathcal{N}_2 \times F^n \times F^n$ of all quadruples (A, B, v, w) such that v is a cyclic vector for $(A, B) \in \mathcal{N}_2$ and w cyclic for (A^T, B^T) is an open subset. The fact that U is open follows since its complement is given by a set of polynomial conditions $\det X = 0$ and $\det Y = 0$, where X runs over all square matrices with columns of the form $A^i B^j v$ and Y over all square matrices with the columns of the form $\left(A^T\right)^i \left(B^T\right)^j w$. Here it is certainly enough to take $0 \le i, j \le n-1$. The same argument shows that also $U_B = \{(A, v, w);\ (A, B, v, w) \in U\}$ is an open subset of $\mathcal{N}_B \times F^n \times F^n$.

Lemma 2.1. *Suppose that $(A, B) \in \mathcal{N}_2$. Then there is a third nilpotent matrix C and vectors $v, w \in F^n$ such that:*

(i) *C commutes with B,*
(ii) *any linear combination $\alpha A + \beta C$ is nilpotent,*
(iii) *v is a cyclic vector for (C, B),*

(iv) *w is a cyclic vector for (C^T, B^T).*

Proof. The proof is an extension of the proof of Baranovsky [1, Lem. 3]. Let $\lambda = \left(\lambda_1^{r_1}, \lambda_2^{r_2} \dots, \lambda_l^{r_l}\right)$, where $\lambda_1 > \lambda_1 > \cdots > \lambda_l > 0$ and $r_i \geq 1$ for all i be the partition corresponding to B. As in the proof of [1, Lem. 3] there is a Jordan basis

$$\{e_{ijk} : 1 \leq i \leq l,\ 1 \leq j \leq r_i,\ 1 \leq k \leq \lambda_i\}$$

for B such that

(1) $Be_{ijk} = e_{ij,k+1}$ if $k < \lambda_i$ and $Be_{ij\lambda_i} = 0$,
(2) Ae_{ijk} is in the linear span of vectors e_{fgh}, where either
  - $f > i$ and g, h arbitrary, or
  - $f = i$ and $g > j$ and h arbitrary, or
  - $f = i$ and $g = j$ and $h > k$.

To simplify our expressions we assume that $e_{ijk} = 0$ if the three indices i, j, k do not satisfy the conditions $1 \leq i \leq l,\ 1 \leq j \leq r_i$ and $1 \leq k \leq \lambda_i$. We also use the difference sequence of λ; we write $\delta_i = \lambda_i - \lambda_{i-1}$ for $i = 2, 3 \dots, l$. Now we define C by $Ce_{ijk} = e_{i,j+1,k}$ if $j < r_i$ and $Ce_{i,r_i,k} = e_{i+1,1,k} + e_{i-1,1,k+\delta_i}$.

To illustrate the actions of B and C we include an example. We consider the case $\lambda = (4^2, 3^2, 2, 1^2)$. To indicate the actions of B and C we draw a directed graph whose vertices correspond to the vectors of the Jordan basis $\{e_{ijk}\}$ and the edges to the nonzero elements of B and C. In the graph in Figure 1 the dashed directed edges correspond to

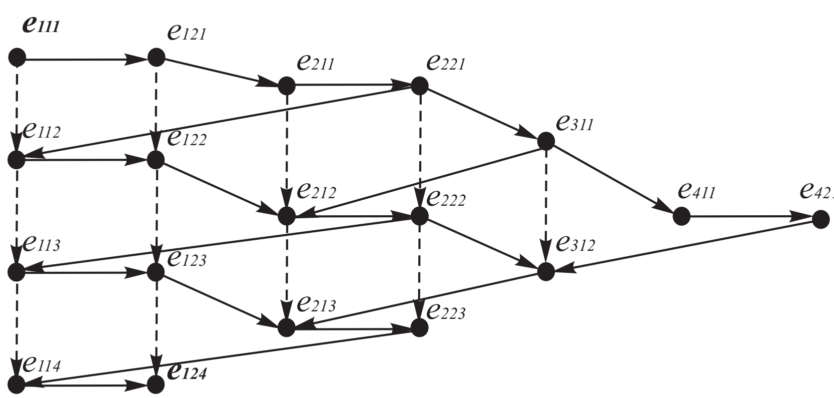

Figure 1.

the action of matrix B and the nondashed ones correspond to the action of matrix C. Note that the graph indicating the action of B^T and C^T is obtained by reversing all the edges in the graph corresponding to B and C. Note that vector e_{111} is cyclic and vector e_{124} cocyclic, i.e., cyclic for B^T and C^T.

Let us continue with the proof. It is easy to show that $BC = CB$; we prove directly that $BCe_{ijk} = CBe_{ijk}$ for all i, j, k. The property (ii) follows from (2) and the definition of C.

Next we show that e_{111} is a cyclic vector for (C, B) and $e_{1r_1\lambda_1}$ is a cyclic vector for (C^T, B^T). We denote by $\mathcal{A}$ the unital subalgebra of $M_n(F)$ generated by B and C and by $\mathcal{A}^T$ the unital subalgebra generated by the transposed matrices B^T and C^T. We show by induction on i that each e_{ijk} is in the subspace $\mathcal{A}e_{111}$. For $i = 1$ we have

$e_{1jk} = B^{k-1}C^{j-1}e_{111}$. We denote by W_i the linear span of all vectors e_{tjk} with $t \leq i$. To prove the inductive step, we have to show that $e_{i+1,j,k} \in \mathcal{A}W_i$. This follows since $e_{i+1,j,k} - B^{k-1}C^j e_{i,r_i,1} \in \mathcal{A}W_i$.

Before we prove that $e_{1r_1\lambda_1}$ is a cyclic vector for (C^T, B^T) observe that $B^T e_{ijk} = e_{ij,k-1}$ and that $C^T e_{ijk} = e_{i,j-1,k}$ if $j > 1$ and $C^T e_{i1k} = e_{i-1,r_{i-1},k} + e_{i+1,r_{i+1},k-\delta_{i+1}}$. Again we proceed to prove that $e_{1r_1\lambda_1}$ is cyclic by induction on i.

For $i = 1$ we have $e_{1jk} = \left(B^T\right)^{\lambda_1-k}\left(C^T\right)^{r_1-j} e_{1r_1\lambda_1}$. The inductive step follows since $e_{i+1,j,k} - \left(B^T\right)^{\lambda_i-k}\left(C^T\right)^{r_i-j} e_{ir_i\lambda_i} \in \mathcal{A}^T W_i$. □

Proposition 2.2. *The subset U is dense in $\mathcal{N}_2 \times F^n \times F^n$ and the subset U_B is dense in $\mathcal{N}_B \times F^n \times F^n$.*

Proof. Consider a quadruple $(A, B, v, w) \in \mathcal{N}_2 \times F^n \times F^n$. By Lemma 2.1 we can find a matrix C and vectors v', w' such that $(C, B, v', w') \in U$. Then the affine line L of all the points $(\alpha A + \alpha' C, B, \alpha v + \alpha' v', \alpha w + \alpha' w')$, where $\alpha \in F$ is arbitrary and $\alpha' = 1 - \alpha$, has nonempty intersection with U. Hence $L \cap U$ is dense in L and (A, B, v, w) is in the closure of U.

The same argument shows that also U_B is dense in $\mathcal{N}_B \times F^n \times F^n$. □

Next we give a proof of a known result which we could not find stated in the literature.

Proposition 2.3. *A commutative subalgebra $\mathcal{R}$ of $M_n(F)$ is Gorenstein if both $\mathcal{R}$ and $\mathcal{R}^T$ have a cyclic vector, i.e., the action of $\mathcal{R}$ is cyclic and cocyclic.*

Proof. By Lemma 2.5, parts (1) and (2), of [13] the subalgebra $\mathcal{R}$ is cyclic if and only if F^n and $\mathcal{R}$ are isomorphic as $\mathcal{R}$-modules, and the subalgebra $\mathcal{R}^T$ is cyclic if and only if F^n and $\mathcal{R}^T$ are isomorphic $\mathcal{R}$-modules. Then our assumptions imply that $\mathcal{R}$ and its dual module are isomorphic $\mathcal{R}$-modules, and thus $\mathcal{R}$ is Gorenstein by the definition [5, pp. 525-526]. □

We denote by $\pi : \mathcal{N}_2 \times F^n \times F^n \to \mathcal{N}_2$ and $\pi_B : \mathcal{N}_B \times F^n \times F^n \to \mathcal{N}_B$ the projections to the first factor. Then, it follows by Proposition 2.3 that $\pi(U)$ is the set of pairs (A, B) of nilpotent matrices such that the unital algebra $\mathcal{A}$ generated by A and B is Gorenstein of (vector space) dimension n. Moreover, such an $\mathcal{A}$ is a complete intersection since its embedding dimension is at most two [5, Cor. 21.20]. Similarly, $\pi_B(U_B)$ is the set of all $A \in \mathcal{N}_B$ such that $\mathcal{A}(A, B)$ is Gorenstein of dimension n.

As a consequence of Proposition 2.2 we obtain the main results of this section:

Corollary 2.4. *The subset $\pi(U)$ of those pairs in $\mathcal{N}_2$ that generate a Gorenstein algebra of (vector space) dimension n is dense in $\mathcal{N}_2$.*

Corollary 2.5. *The subset $\pi_B(U_B)$ is dense in $\mathcal{N}_B$. So, the unital subalgebra $\mathcal{A}$ generated by a nilpotent matrix B and a generic matrix A in $\mathcal{N}_B$ is Gorenstein; moreover, it is a complete intersection. Equivalently, both $\mathcal{A}$ and $\mathcal{A}^T$ have a cyclic vector, i.e., the action of $\mathcal{A}$ is cyclic and cocyclic.*

3. $\mathcal{D}$ is an idempotent map

A pair of commuting nilpotent matrices (A, B) generates a (unital) local artinian algebra $\mathcal{A}$. We denote its maximal ideal by m. The associated graded algebra of $\mathcal{A}$ is

$$\operatorname{gr} \mathcal{A} = \oplus_{i=0}^{k} m^i / m^{i+1},$$

where $m^0 = \mathcal{A}$ and k is such that $m^k \neq 0$ and $m^{k+1} = 0$. The *Hilbert function* $H(\mathcal{A})$ of $\mathcal{A}$ is the sequence $(h_0, h_1, \dots, h_k)$, where

$$h_i = \dim_F m^i / m^{i+1}.$$

We have $\sum_{i=0}^k h_i = \dim_F \mathcal{A}$. If $\mathcal{A}$ is Gorenstein then $h_k = 1$. Furthermore, Macaulay's Theorem on the Hilbert function of the intersection of two plane curves [12] (see Iarrobino [7] or [8, p. 23]) says that the Hilbert function $H(\mathcal{A})$ of an artinian local complete intersection of the embedding dimension at most 2 satisfies

$$H(\mathcal{A}) = (1, 2, \dots, d, h_d, h_{d+1}, \dots, h_i, \dots, h_k),$$

where $h_{d-1} = d \geq h_d \geq h_{d+1} \geq \dots \geq 1$ and $h_{i-1} - h_i \leq 1$ for all $i = d, d+1, \dots, k$ and $h_k = 1$.

In the rest of the paper, we write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 1$. (Compare with the proof of Lemma 2.1, where we used different, 'power type', notation for parts of partition λ.) We assume that $\lambda_j = 0$ for $j > l$.

The set of all partitions $\Lambda(n)$ of n has a partial order given by $\lambda \prec \mu$ if $\sum_{i=1}^j \lambda_i \leq \sum_{i=1}^j \mu_i$ for $j = 1, 2, \dots$. To each partition λ we associate its *Ferrers diagram*, a diagram with λ_i boxes in the i-th row. We denote by $\lambda(H)$ the partition, which has the i-th part equal to the number of elements in the sequence $H = (h_1, h_2, \dots, h_k)$ such that $h_j \geq i$.

Example 3.1. *If $H = (1, 2, 3, 2, 1)$ then $\lambda(H) = (5, 3, 1)$ and if $H = (1, 2, 3, 3, 1)$ then $\lambda(H) = (5, 3, 2)$.*

Here we recall two results of Basili and Iarrobino [3] that we use in the proof below. They proved [3, Thm. 2.21] that the Hilbert functions of the algebras $\mathcal{A}(A, B)$, $A \in \mathcal{N}_B$, determine $\mathcal{D}(\lambda)$:

(1) $$\mathcal{D}(\lambda) = \sup\{\lambda(H(\mathcal{A}));\ \mathcal{A} = \mathcal{A}(A, B),\ A \in \mathcal{N}_B,\ \text{such that } \dim \mathcal{A} = n\}.$$

They also proved [3, Thm. 1.12] that if the parts in a partition λ differ by at least two then λ is stable, i. e. $\mathcal{D}(\lambda) = \lambda$.

The following is the main result of our paper.

Theorem 3.2. *Assume that B is a nilpotent matrix and λ the corresponding partition. Then the partition $\mathcal{D}(\lambda)$ corresponding to the generic element $A \in \mathcal{N}_B$ has decreasing parts differing by at least 2 and $\mathcal{D}(\mathcal{D}(\lambda)) = \mathcal{D}(\lambda)$, i.e. $\mathcal{D}$ is idempotent.*

Proof. Recall that $\mathcal{D}(\lambda)$ is given by (1). Corollary 2.5 implies that for a generic $A \in \mathcal{N}_B$ the algebra $\mathcal{A} = \mathcal{A}(A, B)$ is a complete intersection (therefore Gorenstein) of dimension n. Since $\mathcal{N}_B$ is irreducible and since $\mathcal{D}(\lambda)$ is the partition corresponding to the generic $A \in \mathcal{N}_B$, it is enough to take in (1) the supremum over all $\lambda(H(\mathcal{A}))$, where $\mathcal{A}$ is a complete intersection. The above stated Macaulay's Theorem on the Hilbert function of the intersection of two plane curves then implies that the parts in $\mathcal{D}(\lambda)$ differ by at least two. By [3, Thm. 1.12] or [9, Cor 1.5] it follows that $\mathcal{D}(\mathcal{D}(\lambda)) = \mathcal{D}(\lambda)$. □

4. Description of $\mathcal{D}(\lambda)$ when it has at most two parts

A partition λ is called a *almost rectangular* if the largest and the smallest part of λ differ by at most one. (Note that in [3] the term *string* is used for almost rectangular partition.) We denote by r_B or r_λ the smallest number r such that λ is a union of r almost

rectangular partitions. (Here B is a nilpotent matrix such that its Jordan canonical form is determined by λ.) Basili [2, Prop. 2.4] proved that a generic $A \in \mathcal{N}_B$ has rank $n - r_B$, which is maximal possible in $\mathcal{N}_B$ and that $\mathcal{D}(\lambda)$ has r_B parts.

Example 4.1. *For instance, if* $\lambda = (4, 3, 2, 1)$ *and* $\mu = (7, 7, 6, 4, 4, 3, 2)$ *then* $r_\lambda = 2$ *and* $r_\mu = 3$.

It is an interesting and important question to describe $\mathcal{D}(\lambda)$ in terms of λ (see [18, §3], in particular [18, Problem 1]). Using [14, Thm. 13] and [4, Cor. 3.29] it is easy to answer this question when $\mathcal{D}(\lambda)$ has at most two parts, i. e. when $r_B \le 2$. Here we would like to remark that the proofs of Lemma 11 and Theorems 12 and 13 in [14] hold over any field of characteristic 0, while Basili and Iarrobino assume in [4] that the underlying field is algebraically closed of arbitrary characteristic.

Now, if $r_B = 1$ then $\mathcal{D}(\lambda) = (n)$. If $r_B = 2$ then it is enough to know the maximal index of nilpotency of an element of $\mathcal{N}_B$ to describe $\mathcal{D}(\lambda)$, since $\mathcal{D}(\lambda)$ is the maximal partition among those corresponding to elements $A \in \mathcal{N}_B$ [3, Lem. 1.6]. Applying the description of the maximal index of nilpotency given in [14, Thm. 13] and [4, Cor. 3.29], we have the following result.

Theorem 4.2. *If* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_l \ge 1$ *is such that* $r_\lambda = 2$ *then* $\mathcal{D}(\lambda) = (i_\lambda, n - i_\lambda)$, *where* i_λ *is the maximal index of nilpotency in* $\mathcal{N}_B$ *given by*

$$i_\lambda = \max_{1 \le i < l} \left\{ 2(i-1) + \lambda_i + \lambda_{i+1} + \ldots + \lambda_{i+r};\ \lambda_i - \lambda_{i+r} \le 1,\ \lambda_{i-1} \ge 2 \text{ if } i > 1 \right\}.$$

Example 4.3. *Suppose that* $\lambda = (4, 4, 3, 3, 2)$ *and* $\mu = (5, 5, 3, 3, 2)$. *Then we have* $\mathcal{D}(\lambda) = (14, 2)$ *and* $\mathcal{D}(\mu) = (12, 6)$.

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